Numerical studies on the dynamics of the globally coupled maps with sequential updating

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A system of globally coupled logistic maps with sequential updating is analyzed numerically. It is found that deterministic asynchronous updating schemes may have dramatical influences on the dynamical behaviors of globally coupled systems. Transitions from spatiotemporal chaos to spatially organized states are observed as the coupling parameter varies. It is shown that the model system may exhibit a variety of collective properties such as clustering, traveling wave patterns, and spatial bifurcation cascades.

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The study of large assemblies of chaotic elements that can spontaneously evolve to a state of large scale synchronization is important not only for understanding nonlinear dynamical systems with many degrees of freedom, but also from the viewpoint of potential applications in biological information processing, evolutionary dynamics, and economics [1–4].

The globally coupled map (GCM), introduced by Kaneko [5], exhibits certain interesting dynamical properties such as clustering of synchronization, suppression of chaos, etc., which are often observed in many real globally coupled systems. However, in many mathematical models for globally coupled systems, the updating mechanism is assumed to be synchronous, while in real neural networks, updating mechanisms are, in some occasions, asynchronous. It is, therefore, interesting to investigate what influences the asynchronous updating schemes may have on the collective behaviors of the globally coupled systems. Recently, a stochastic asynchronous updating mechanism was introduced into the usual GCM [5]. The clustering phenomenon is eliminated due to the element-dependent global feedback onto the individual elements. An inverse bifurcation cascade with coupling strength is observed. It is argued that the disappearance of clustering and the suppression of chaos is due to the internal noises induced by the stochastic asynchronous updating.

In this paper, we study the collective behavior of an ensemble of globally coupled logistic maps with sequential updating through numerical simulations. By sequential updating, we mean that at each iteration step, the state of the individual elements is updated according to a given sequence and the order of the elements in this sequence is fixed during the evolution. Our model system may be described by the following GCM:

$$x^i_{n+j/N}=\begin{cases}(1-\epsilon)f(x^i_n)+\epsilon F^j_n, & \text{if } i=j\\ x^i_n, & \text{if } i\neq j\end{cases},\tag{1}$$

where ϵ is the coupling constant, n is a discrete time step, and i is the index of an element ($i=1,2,\ldots,N=$ system size). The logistic map $[f(x)=1-ax^2]$, is chosen as a prototype for a system of globally coupled chaotic system. The sequential updating scheme is then defined by the following global feedback:

$$F^i_n=\frac{1}{N}\left[\sum_{k=1}^{i-1} f(x^k_{n+1})+\sum_{k=i}^{N} f(x^k_n)\right].\tag{2}$$

It is noticed that, by Eq. (1), the feedback term (2) contains both global and local information. It also introduces an updating wave that travels along the system at a velocity of one site per iteration. This kind of model system emphasizes the importance of the finite propagation velocity of the interaction, assuming that the reaction is instantaneous. The system has certain intrinsic spatial inhomogeneity.

We have performed extensive numerical simulations on the globally coupled logistic maps with sequential updating. We find that the different distributions of random initial conditions may lead to qualitatively distinct dynamical behavior of the model system under consideration. For the convenience of simplicity, we focus our attention on the uniform distributions of initial conditions, i.e., the amplitude of each site is chosen randomly from the interval ($-\eta<x^i_0<\eta$) ($i=1,2,\ldots,N$) with $0<\eta\leqslant 1$. It is possible that more diversified dynamical behaviors can be found if the different element is allowed to have a distinct distribution for its initial condition.

In Fig. 1 we show the bifurcation diagrams for the evolution of a single element as a function of ϵ, for $N=2000$ and $a=1.8$. Figure 1(a) shows the evolution of a single element as a function of the coupling constant ϵ for $\eta=1$. It is seen that this result is quite similar to that found in Ref. [6]. Note that here we have used the same initial conditions for all ϵ, which leads to well-defined bifurcation branches. Otherwise small dispersions in the bifurcation branches may be observed due to the presence of coexisting multistable states that have only quantitative differences in their amplitudes. It is obvious that the coupled system may have many coexist-

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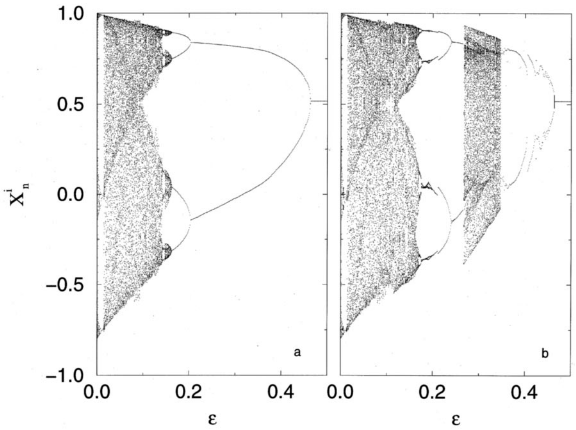

FIG. 1. Bifurcation diagrams for a system of $N=2000$ logistic maps, as a function of the coupling constant ϵ. The diagrams are constructed from the evolution of a randomly chosen single map, with $a=1.8$ for (a) $\eta=1$ and (b) $\eta=0.67$. The same initial conditions are used for each value of ϵ in order to reduce the fluctuation due to the coexisting multistable attractors.

ing multistable states. However, for our model system the number of quantitatively different coexisting multistable attractors is reduced as the system size N is increased. Our numerical results reveal that qualitatively different attractors of the coupled system may coexist for systems with small sizes.

There are several crucial differences between our model system and that investigated in Ref. [6]. First, our system is deterministic and spatially inhomogeneous, while the latter system has intrinsic noise and is spatially homogeneous, at least statistically. In view of the spatial inhomogeneity, it is clear that a different single element may exhibit relatively different bifurcation scenarios. Second, in all previously studied globally coupled systems the global feedback is in principle a mean-field quantity, while in our system it is not. This difference is responsible for some unusual properties observed only in our model system and makes the mean-field-like analysis difficult and even meaningless. Figure 1(b) shows much richer structures of the bifurcation diagram when $\eta<1$. In this case, as the coupling constant ϵ is increased one may observe the following phases. (i) Spatiotemporal chaos. In this regime, since the coupling is quite weak, each site behaves almost independently. (ii) Spatial frozen patterns. Here the system is spatially inhomogeneous, and different elements may possess different dynamical properties. But because of sequential updating there exist certain relations among neighboring site. In fact, spatial bifurcations are observed. (iii) Traveling patterns. The coupling in this regime is strong enough to establish the spatial homogeneity. Elements at different positions are related by constant phase differences. (iv) Periodic orbits. The spatiotemporal chaos is suppressed by the particular feedback mechanism (2). To better illustrate the quantitative changes in the dynamical behavior as ϵ varies, in Fig. 2 we plot the phase diagrams corresponding to two different initial conditions characterized by η. Each phase is calculated by summarizing the behavior of a single element as a and ϵ vary. For each value of a (the logistic map parameter) and ϵ the evolution is calculated for 30 000 time steps. It should be

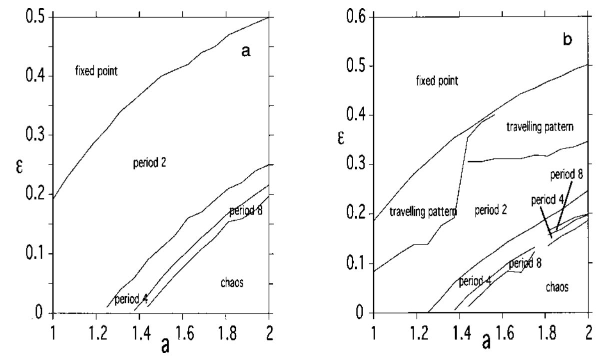

FIG. 2. The phase diagrams of our model system with (a) $\eta=1$ and (b) $\eta=0.67$. Phases are determined by the typical temporal behaviors of a single element chosen from the coupled system of totally $N=1000$ maps.

stressed that the attractors displayed in the phase diagrams are, in principle, attractors of a single element.

In Fig. 2(a), we plot the variety of the dynamical behaviors for $\eta=1$. It is seen that there exists a well-defined boundary between the fixed point and period-2 zones, which turns to the same boundary line defined in Ref. [6]. Figure 2(b) exhibits the possible phases for $\eta=0.67$. A traveling pattern phase appears for sufficiently large global coupling constant ϵ. In fact, this phase is already present in Fig. 2(a), but with small basins. When the system operates in this regime, each element of the coupled system is dynamically equivalent with a finite time lag among the different elements. It is found that the velocity of the propagating patterns is a function of ϵ, which is smaller than the velocity of the updating. When further increasing ϵ, the system falls on two-cluster period-2 attractors, which is a typical feature of globally coupled systems, though in our case, the cluster is defined only through the phases of the constituents.

Figure 3(a) shows typical spatial bifurcation cascades corresponding to the spatial frozen phase, in which the the spatial homogeneous is broken and different elements may show qualitatively different dynamics. If we characterize the elements according to their periods or frequencies, then in the regime of frozen spatial pattern we find that the clusters of different periods are formed, which are organized to exhibit a spatial bifurcation pattern, resulting from the interplay of global coupling and sequential updating. The clustering is a typical dynamical behavior in globally coupled systems, in which spatial inhomogeneity is a direct consequence of the asynchronous updating mechanism. The rather different dynamical features displayed by neighboring elements is in contrast with the intuition that on average the difference of the global feedback F_n^i on neighboring sites is typically of order $2/N$, which is very small for N large. Note that the elements in a periodic cluster can be further divided into groups, each of them consists of those elements that are on the same periodic branch. To analyze the complicated dynamics it is natural to expect that such a spatial bifurcation may arise from a spatial modulation in the global feedback. We have also examined the behavior of the global feedback $F(x_n^i)$ and the term $(1-\epsilon)f(x_n^i)$. It is found that the essential dynamical features are characterized by the term $(1-\epsilon)f(x_n^i)$, and $F(x_n^i)$ only provides some quantitative modi-

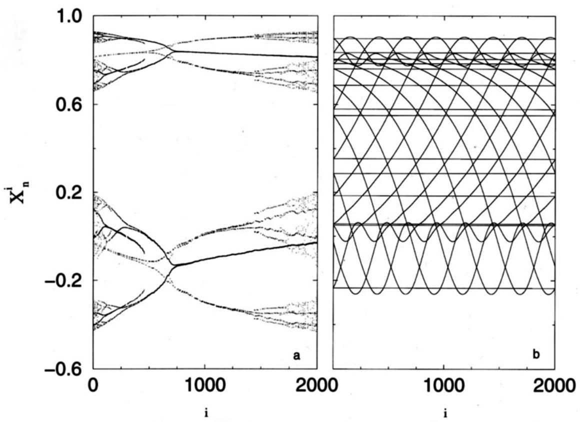

FIG. 3. The frozen pattern and the traveling pattern displayed by the globally coupled maps with sequential updating. The system parameters are given by $N=2000$, $a=1.8$, and $\eta=0.67$. The amplitudes of the elements x_n^i are plotted as a function of space index i for (a) $\epsilon=0.2$, and (b) $\epsilon=0.3$.

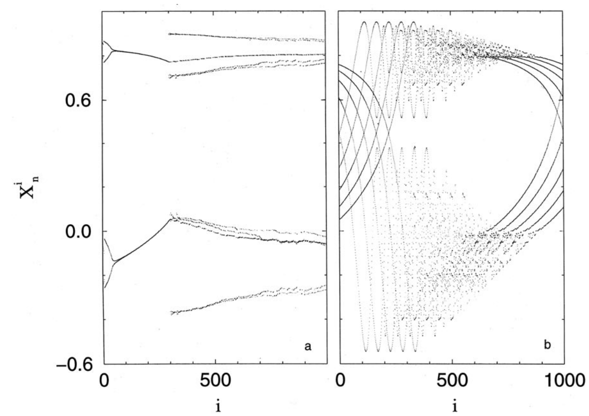

FIG. 5. The interaction between two subsystems with different types of distributions for initial conditions. The system parameters are $N=1000$, $a=1.9$, and $\epsilon=0.25$. (a) $N_1=300$; (b) $N_1=400$. The critical value $N_c=282$.

fications. This fact suggests that one cannot take $F(x_n^i)$ as a global feedback and develops a mean-field-like approach to explain the dynamical properties of our model system.

Another typical dynamical feature of our model system is the existence of traveling patterns, which is displayed in Fig. 3(b). In this regime the spatial translation symmetry is restored. The detailed properties of traveling waves such as the velocity, wave form, etc., depend on the system parameters a and ϵ. For fixed a, the propagation velocity is proportional to the global coupling constant $\epsilon(a)$. It is found that for $\epsilon > \epsilon_c$, the traveling wave begins to form and move along the direction of the updating. The elements of the system are strongly correlated with finite phase differences among them. The temporal behavior changes from the breathing waves to rotating ones, as ϵ varies within the parameter range of the traveling patterns. For ϵ sufficiently large, the system is driven into a period-2 cluster state with an appreciable fluctuation among the amplitudes of individual elements. In Fig. 3(b) we plot these dynamical states for $a=1.8$, $\epsilon=0.2$, and $\eta=0.67$. Here again we find that the main dynamical properties come from the the term $(1-\epsilon)f(x_n^i)$.

The dependence of the dynamical behavior on the choice of the distribution of the initial conditions is remarkable. For example, in Fig. 4 we show a breathing pattern for $a=1.8$, $\epsilon=0.19$, and $\eta=0.21$. A phase transition from static patterns to traveling ones takes place at $\epsilon_c=0.187$, approximately. The sensitivity of the system to the initial conditions is exposed by examining a composed system consisting of two parts, each one starting with different initial conditions. For illustration, we use two extreme distributions of initial conditions. One is the constant initial condition, i.e., $x_0^i=x_0$ for all elements. The other is the uniform distribution with $\eta=1$, i.e., $-1<x_0^i<1$. The system parameters are taken to be $N=1000$, $a=1.9$, $\epsilon=0.25$, and $N_1=300$ in Fig. 5(a) and $N_1=400$ in Fig. 5(b). As shown previously, the subsystem S_1 originally (i.e., $N_1=N$) has a traveling solution, while the subsystem S_2 has a periodic solution. Although there still exists the boundary that divides the two subsystems, the resultant dynamics is distinct from both the original S_1 and S_2, in Fig. 5(a). In Fig. 5(b) we see that the subsystem S_2 is synchronized with S_1.

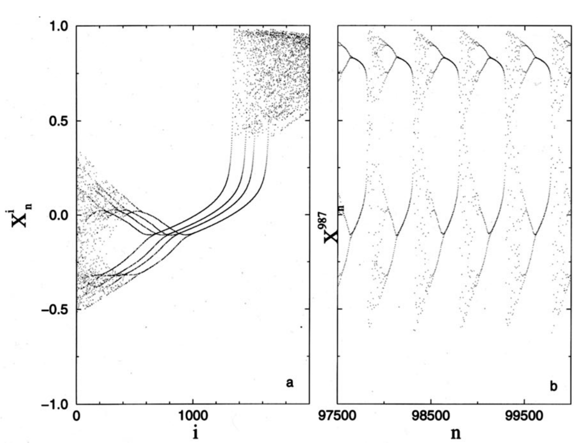

FIG. 4. The breathing patterns for $N=2000$, $a=1.8$, $\epsilon=0.19$, and $\eta=0.21$. (a) x_n^i vs i, and (b) the temporal behavior of an arbitrarily chosen element $i_0=987$.

The findings in this work may have some implications on the information processing in neural networks. It is known that the processing of information is scattered among different areas of the brain and the information may be linked through synchronous activities, using temporal codes. Neural networks that are more realistic from the physiological viewpoint, and that make use of synchronous oscillatory behavior, have been introduced in the context of visual processing [7,8]. That brain activity might be shaped by deterministic chaos has been suggested by Freeman [9] and studied by using an assemblies of chaotic model neurons in Ref. [10]. From our numerical simulations, it is observed that the elements may be organized to form groups with different dynamical behaviors, which may be used to perform different tasks in the information processing. Obviously, our findings only suggest a possible way of the formation of clusters with different dynamical features. In order to capture the essential properties of information processing of the neuronal systems, one has to use more realistic neuron models.

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